Appendix G

Integration by parts in two or three dimensions (Green’s theorem)

Consider the integration by parts of the following two-dimensional expression

\[ \int \int_{\Omega} \phi \frac{\partial \psi}{\partial x} \, dx \, dy \]  

\( \text{(G.1)} \)

Integrating first with respect to \( x \) and using the well-known relation for integration by parts in one dimension

\[ \int_{x_L}^{x_R} u \, dv = - \int_{x_L}^{x_R} v \, du + (uv)_{x=x_R} - (uv)_{x=x_L} \]  

\( \text{(G.2)} \)

we have, using the symbols of Fig. G.1,

\[ \int \int_{\Omega} \phi \frac{\partial \psi}{\partial x} \, dx \, dy = - \int \int_{\Omega} \frac{\partial \phi}{\partial x} \psi \, dx \, dy + \int_{y_T}^{y_B} [(\phi \psi)_{x=x_R} - (\phi \psi)_{x=x_L}] \, dy \]  

\( \text{(G.3)} \)

If now we consider a direct segment of the boundary \( d\Gamma \) on the right-hand boundary, we note that

\[ dy = n_x \, d\Gamma \]  

\( \text{(G.4)} \)

where \( n_x \) is the direction cosine between the outward normal and the \( x \) direction. Similarly on the left-hand section we have

\[ dy = -n_x \, d\Gamma \]  

\( \text{(G.5)} \)

The final term of Eq. (G.3) can thus be expressed as the integral taken around an anticlockwise direction of the complete closed boundary:

\[ \oint_{\Gamma} \phi \psi n_x \, d\Gamma \]  

\( \text{(G.6)} \)

If several closed contours are encountered this integration has to be taken around each such contour. The general expression in all cases is

\[ \int \int_{\Omega} \phi \frac{\partial \psi}{\partial x} \, dx \, dy = - \int \int_{\Omega} \frac{\partial \phi}{\partial x} \psi \, dx \, dy + \oint_{\Gamma} \phi \psi n_x \, d\Gamma \]  

\( \text{(G.7)} \)
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Similarly, if differentiation in the \( y \) direction arises we can write

\[
\iint_{\Omega_1} \phi \frac{\partial \psi}{\partial y} \, dx \, dy = - \iint_{\Omega_1} \frac{\partial \phi}{\partial y} \psi \, dx \, dy + \oint_{\Gamma_1} \phi \psi n_y \, d\Gamma
\]

where \( n_y \) is the direction cosine between the outward normal and the \( y \) axis.

In three dimensions by identical procedure we can write

\[
\iiint_{\Omega_1} \phi \frac{\partial \psi}{\partial y} \, dx \, dy \, dz = - \iiint_{\Omega_1} \frac{\partial \phi}{\partial y} \psi \, dx \, dy \, dz + \oint_{\Gamma_1} \phi \psi n_y \, d\Gamma
\]

where \( d\Gamma \) becomes the element of the surface area and the last integral is taken over the whole surface.